CANTOR SYSTEMS
DIMENSION GROUPS
AND BRATTÉLÍ DIAGRAMS

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Contents

Introduction 3

1 Cantor systems 4
   1.1 Dynamical systems ............................................. 4
   1.2 Minimal dynamical systems ..................................... 4
   1.3 Cantor sets and systems ...................................... 5

2 Simple ordered Bratteli diagrams 6
   2.1 Definition and properties of simple ordered Bratteli diagrams .... 6
   2.2 Cantor systems generated by simple ordered Bratteli diagrams ...... 9
   2.3 Model theorem for Cantor systems ................................ 13

3 Dimension groups 14
   3.1 Ordered Abelian groups ......................................... 14
   3.2 Definition of dimension groups .................................. 15

4 The simple dimension group \( K^0(X,T) \) 18
   4.1 Definition of \( K^0(X,T) \) ........................................... 18
   4.2 Equivalence and comparison criteria for the functions in \( C(X,Z) \) ... 20
   4.3 \( K^0(X,T) \) is a simple dimension group ....................... 22
   4.4 Characteristic functions ......................................... 24
   4.5 The full group .................................................. 24

5 Measures and integration on Cantor systems 26
   5.1 Measures on Cantor sets ......................................... 26
   5.2 Invariant measures on Cantor systems .......................... 27
   5.3 Measures as positive states on \( K^0(X,T) \) ....................... 31
   5.4 Characterisation of positive \( K^0 \) elements by integrals .............. 32
   5.5 Example of measure with only trivial invariant transformation .......... 33

Index 35

Bibliography 37
Introduction

Recent results in topological dynamics, described in [3], have shown that ordered cohomology is the key to investigate the orbit structure of Cantor minimal dynamical systems. The establishment of these results requires the use of methods from homological algebra and $K$-theory.

Similarly, the full group of homeomorphisms of a Cantor system also gives information about its orbit structure.

An important tool in all this is the recent Bratteli-Vershik model theorem (here theorem 2.7), which allows Cantor minimal dynamical systems to be investigated using the relatively concrete structures of simple ordered Bratteli diagrams.

We will here use the model theorem to give new and simplified proofs of some key results related to the naturally defined $K^0$-group of a Cantor system. In particular we prove that the $K^0$-group of a Cantor system is a simple dimension group. We also investigate states on this group, or equivalently measures on the Cantor set invariant under the dynamics of the system, and show how the ordering of the $K^0$-group is determined by these.

In addition, we show a theorem useful for the construction of elements of the full group.

At last, we briefly consider the process of finding dynamical systems leaving a given measure invariant, constructing an example of a non-atomic probability measure having no nontrivial such system, effectively killing the possibility of characterizing orbit equivalent Cantor minimal systems solely in terms of non-atomic probability measures (cf. [3, theorem 2.2]).
Chapter 1

Cantor systems

1.1 Dynamical systems

By $\mathbb{Z}$ we shall mean the set of integers, by $\mathbb{Z}^+$ the set of nonnegative integers, by $\mathbb{R}$ the set of real numbers and by $\mathbb{R}^+$ the set of nonnegative real numbers.

Given a space $X$ we shall let $\text{Id}_X$ denote the identity function on $X$, i.e. $\text{Id}_X : X \to X, x \mapsto x$.

By a (topological) dynamical system we shall here mean a pair $(X, T)$, where $X$ is a compact Hausdorff space and $T : X \to X$ is a homeomorphism.\footnote{In the general literature, $X$ need not be compact, and $T$ may be replaced by a topological group of homeomorphisms on $X$.}

There is a natural notion of isomorphism between topological dynamical systems known as conjugacy, defined as follows: $(X_1, T_1)$ is conjugate to $(X_2, T_2)$ if there is a homeomorphism $\phi : X_1 \to X_2$ such that $T_2 \circ \phi = \phi \circ T_1$. $\phi$ is then a conjugacy isomorphism, or a conjugacy between the two dynamical systems.

Let $(X, T)$ be a topological dynamical system, and let $x \in X$. We will let $T^n$ denote iterations of $T$ for $n \in \mathbb{Z}^+, n > 0$; $T^0 = \text{Id}_X$ and $T^{-n} = (T^{-1})^n$ for $n < 0$. makes the set $\{T^n|n \in \mathbb{Z}\}$ with composition into an Abelian group, and $n \mapsto T^n$ into a group homomorphism from $\mathbb{Z}$ to this group.

By the orbit of $x$ we shall mean the set $\{T^n(x)|n \in \mathbb{Z}\}$. By the positive orbit of $x$ we shall mean the set $\{T^n(x)|n \in \mathbb{Z}^+\}$. By the (positive) orbit of a set $M \subset X$ we shall mean the union of the (positive) orbits of the points of $M$. There is also the concept of negative orbits, which is equivalent to positive orbits of $(X, T^{-1})$.

1.2 Minimal dynamical systems

The following is a well known theorem of topological dynamics:

**Theorem 1.1** Let $(X, T)$ be a topological dynamical system. The following are equivalent:
1. For every open set $O$ of $X$, if $T(O) = O$ then $O = \emptyset$ or $O = X$.
2. For every closed set $F$ of $X$, if $T(F) = F$ then $F = \emptyset$ or $F = X$.
3. For every open set $O$ of $X$, if $T(O) \subset O$ then $O = \emptyset$ or $O = X$.
4. For every closed set $F$ of $X$, if $T(F) \subset F$ then $F = \emptyset$ or $F = X$.
5. For every open set $O$ of $X$, if $T(O) \supset O$ then $O = \emptyset$ or $O = X$.
6. For every closed set $F$ of $X$, if $T(F) \supset F$ then $F = \emptyset$ or $F = X$.
7. For every point $x \in X$, the orbit of $x$ is dense in $X$.
8. For every point $x \in X$, the positive orbit of $x$ is dense in $X$.
9. For every nonempty open subset $O$ of $X$, the orbit of $O$ equals $X$.
10. For every nonempty open subset $O$ of $X$, the positive orbit of $O$ equals $X$.

A proof of the theorem can be found e.g. in [7]. Here we note that (1) $\iff$ (2), (3) $\iff$ (6), and (4) $\iff$ (5) are obvious since open and closed sets are complements; (10) $\Rightarrow$ (9) and (8) $\Rightarrow$ (7) are also obvious.

A dynamical system which fulfils the criteria in theorem 1.1 we call minimal.

1.3 Cantor sets and systems

A Cantor set is a compact metrizable Hausdorff space without isolated points which has a countable basis of clopen sets, i.e. sets that are both open and closed.

It is a well known theorem (see, e.g. [8]) that all Cantor sets are homeomorphic.

**Example 1.1** Consider the subset of $[0, 1]$ obtained by removing the open middle third of this interval, then the open middle third of the remaining subintervals, and so on, and letting the Cantor set be the closed set which is the intersection of all the remaining subsets. See figure 1.1.

We define a Cantor system as a minimal topological dynamical system $(X, T)$ where $X$ is a Cantor set. Up to conjugacy it does not matter which particular Cantor set we use, as all Cantor sets are homeomorphic and any homeomorphism between two Cantor sets can be used as a conjugacy to transfer a dynamical system from one to the other.
Chapter 2

Simple ordered Bratteli diagrams

2.1 Definition and properties of simple ordered Bratteli diagrams

A Bratteli diagram is a quadruple \((V, E, r, s)\) such that: \(V\) and \(E\) are the vertex set and the edge set, respectively; and \(r, s : E \to V\), known as the range map and source map, respectively, uniquely define the sets \(V_i, i \in \mathbb{Z}^+\), and \(E_i, i \geq 1, i \in \mathbb{Z}^+\), with the properties:

1. \(\bigcup_{i=0}^{\infty} V_i = V\) as a disjoint union;
2. \(\bigcup_{i=1}^{\infty} E_i = E\) as a disjoint union;
3. \(V_0\) is a one point set, \(\{v_0\}\);
4. \(s(E_{i+1}) = V_i\) for each \(i \in \mathbb{Z}^+\);
5. \(r(E_i) = V_i\) for each \(i \geq 1, i \in \mathbb{Z}^+\);
6. \(V_i\) and \(E_i\) are finite sets for each \(i \geq 1, i \in \mathbb{Z}^+\).

The elements of \(V\) are called vertices, and the elements of \(E\) are called edges. We will usually denote the entire diagram by just \((V, E)\) when there is no risk of confusion.

The level of a vertex \(v\) or edge \(e\) is the unique integer \(\text{lev}(v) \geq 0\) or \(\text{lev}(e) \geq 1\) such that \(v \in V_{\text{lev}(v)}\) or \(e \in E_{\text{lev}(e)}\).

Given a Bratteli diagram. We define a path \(p\) in \((V, E)\) to be a sequence \((p_i)_{i=1}^{N}\) of elements from \(E\), such that \(r(p_i) = s(p_{i+1})\) whenever \(i \in \mathbb{Z}^+, 1 \leq i < N\), where \(N \in \mathbb{Z}^+ \cup \{\infty\}\) is the length of the sequence, \(\infty\) denoting a sequence of infinite length. The source of the path is defined as \(s(p) = s(p_1)\), and the range of the path is defined as \(r(p) = r(p_N)\) when the length \(N\) of the path is finite.

There is a canonical 1-1 correspondence between edges and paths of length 1, and we will ignore the distinction when no confusion arises.
If \( v_1 \) and \( v_2 \) are vertices, we may define the set of paths from \( v_1 \) to \( v_2 \) as \( P(v_1, v_2) = \{ p \mid p \text{ path}, s(p) = v_1, r(p) = v_2 \} \). We will replace \( v_1 \) and/or \( v_2 \) by sets of vertices when appropriate.

The **incidence matrix** between level \( i - 1 \) and \( i \) is defined as the \( V_i \times V_{i-1} \) matrix \( M_i \) given by \( M^i_{v_2, v_1} = |P(v_1, v_2)| \) where \( v_1 \in V_{i-1}, v_2 \in V_i \) and \( |P(v_1, v_2)| \) denote the **cardinality** of \( P(v_1, v_2) \).

We denote the **matrix product** of a sequence \((A_i)_{i=k}^{l}\) of matrices by \( \prod_{i=k}^{l} A_i \).

**Theorem 2.1** If \( v_1 \in V_{i_1} \) and \( v_2 \in V_{i_2}, i_1, i_2 \in \mathbb{Z}^+, i_1 \leq i_2 \) then \( \prod_{j=i_1+1}^{i_2} M_j \) is a partially ordered Bratteli diagram \((V, E, <, r, s)\) such that \( P(v_1, v_2) \) is a partial order on \( E \) such that \( e_1, e_2 \in E, e_1 \neq e_2 \) are comparable if and only if \( r(e_1) = r(e_2) \). Since each \( E_i \) is finite, each set of mutually comparable edges is also finite, and so each edge with a given range can be uniquely assigned the natural number denoting its place in the order.

In figure 2.1 we show a standard picture of part of an ordered Bratteli diagram: Each vertex is shown as a dot, and each edge as a line segment between its source and range, with its ordinal number written beside it (except when all edges with the same range have the same source.)

The ordering of the edges induces a canonical total order on each of the path sets \( P(v_0, v) \) where \( v \) is a vertex, defined as \( p < q \) if \( p \neq q \), \( p_n < q_n \) for some \( n, n \in \mathbb{Z}^+, 1 \leq n \leq \text{lev}(v) \) and \( p_i = q_i \) for all \( i \) such that \( n < i \leq \text{lev}(v) \). We denote the **successor** and **predecessor** of a path \( p \) in this ordering by \( \text{succ}(p) \) and \( \text{pred}(p) \), respectively.

A **minimal path** is a path such that all of its edges are minimal in the ordering; similarly for **maximal paths**. We see that a path in \( P(v_0, v) \), \( v \in V \), is minimal (maximal) if and only if it is the unique minimal (maximal) path in \( P(v_0, v) \) with respect to the induced ordering.

By a Zorn’s lemma argument it is easy to see that there must exist minimal and maximal infinite paths, which need not be unique, however.

Let \( 0 = n_0 < n_1 < \ldots < n_i < \ldots \) be an infinite sequence of positive integers. The **telescoping** of \( (V, E, <) \) to \( (n_i) \) is the ordered Bratteli diagram \((V', E', r', s', <')\) given by:

1. \( V' = \bigcup_{i=0}^{\infty} V_{n_i} \)

\( \square \)
Figure 2.1: Example of an ordered Bratteli diagram
2. \( E' = \bigcup_{i=0}^{\infty} P(V_n, V_{n+1}) \)

3. \( r', s' \), and \(<' \) are \( r, s \), and \(< \), respectively, applied to paths in \( E' \).

By telescoping to the sequence \( 0, n, n+1, n+2, \ldots \) we see that the paths \( P(v_0, V_n) \) of the original ordered Bratteli diagram become top edges of the telescoped diagram. When drawing parts of such a telescoped diagram we will sometimes omit the vertex \( v_0 \) because otherwise the drawing will become too cramped due to the large number of edges.

By removing in the above definition all references to ordering, we get a telescoping of a Bratteli diagram. We then have the following definition:

**Definition 2.1** A simple Bratteli diagram is a Bratteli diagram \((V, E)\) such that:

1. \((V, E)\) has a telescoping that is completely connected, i.e. in the telescoping any two vertices at consecutive levels are connected by at least one edge;
2. \((V, E)\) is nontrivial i.e. there is an infinite number of levels \( n \) such that \( E_n \) contains two or more edges.

The following alternative criterion allows us to shorten some of our proofs:

**Theorem 2.2** A Bratteli diagram is simple if and only if it has a telescoping such that in the telescoping any two vertices at consecutive levels are connected by at least \( 2 \) edges;

**Proof:** One way is obvious. To prove the other way, we first telescope to a completely connected diagram. We then find a sequence of levels \( 0 < n_1 < \ldots < n_i < \ldots \) such that there is more than one edge to level \( n_i \) in the Bratteli diagram for each \( i \in \mathbb{Z}^+ \). Telescoping to \( 0, n_1, \ldots, n_i, \ldots \) then gives the result. \( \square \)

**Definition 2.2** A simple ordered Bratteli diagram is an ordered Bratteli diagram \((V, E, <)\) such that:

1. \((V, E)\) is simple;
2. There is a unique minimal infinite path \( x_{\text{min}} \) with \( s(x_{\text{min}}) = v_0 \) and a unique maximal infinite path \( x_{\text{max}} \) with \( s(x_{\text{max}}) = v_0 \).

See figure 2.2 for an example.

**2.2 Cantor systems generated by simple ordered Bratteli diagrams**

Let \((V, E)\) be a simple Bratteli diagram. Let \( X \) be the set of infinite paths with source \( v_0 \). We will give a topology on \( X \) which makes it a Cantor set.
Figure 2.2: Simple ordered Bratteli diagrams have a completely, multiply connected telescoping
Let \( p \) be any path in \( P(v_0, v) \) for some \( v \in V \). Then the **cylinder set** of \( p \) is the set \( \tilde{p} = \{ x \in X | p \text{ is a subpath of } x \} \).

Define \( B_i = \{ \tilde{p} | p \in P(v_0, V_i) \} \). Since each infinite path must have a unique subpath between every pair of levels, \( B_i \) is a partition of \( X \) for each \( i \). Moreover, if \( i < j \), then each cylinder set in \( B_j \) is a subset of a cylinder set in \( B_i \), so \( B_i \) is a coarser partition than \( B_j \).

We take the topological basis of \( X \) to be the union of the \( B_i \). By the **level** of a basis element \( M \) we denote the least \( i \) such that \( M \in B_i \).

**Theorem 2.3** \( X \) with the topology given by \( B \) is a Cantor set.

**Proof:** Each element of each \( B_i \) is the complement of the union of the other elements and thus closed, so this basis is clopen. Each \( B_i \) is finite, so the basis is countable.

Assume ad absurdum that \( x \) is an isolated point of \( X \), so \( \{ x \} \) is open; Choose \( i \in \mathbb{Z}^+ \), \( v \in V_i \), \( p \in P(v_0, v) \) such that \( \tilde{p} = \{ x \} \). By theorem 2.2 choose \( 0 = n_0 < n_1 < \ldots < n_j < \ldots \) to be a sequence of completely, multiply interconnected levels of \((V, E)\). Pick \( n_j > i \);

Let \( q \) be a path from level \( n_{j+1} \) to \( n_{j+2} \) which is not a subpath of \( x \). Choose \( y \) as an infinite path containing both \( p \) and \( q \) as subpaths (which is possible because levels \( n_j \) and \( n_{j+1} \) are completely connected.) Now \( y \in \tilde{p} \) but \( y \neq x \) and we have a contradiction. Thus \( X \) has no isolated points.

We shall define a metric \( d \) compatible with the topology of \( X \). Let \( x \neq y \) be paths of \( X \); then let \( d(x, y) = 2^{-\min\{l \in \mathbb{Z}^+ | x_l \neq y_l \}} \), and let \( d(x, x) = 0 \). Clearly \( d(x, y) = 0 \iff x = y \) and \( d(x, y) = d(y, x) \); Now, for \( x, y, z \in X \), all distinct, \( \min\{l \in \mathbb{Z}^+ | x_l \neq y_l \} \leq \min\{l \in \mathbb{Z}^+ | x_l \neq z_l \} \) and so for general \( x, y, z \in X \), \( d(x, z) \leq d(x, y) + d(y, z) \).

So \( d \) is a metric. Assume \( x \in \tilde{p} \), \( p \in P(v_0, v_n) \). Then \( \{ y \in X | d(x, y) < 2^{-n} \} = \tilde{p} \). So the open balls of the \( d \) metric generate the same topology as the \( B_i \).

To see that \( X \) is compact, assume to the contrary that \( (x_i)_{i \in \mathbb{Z}^+} \) is a sequence in \( X \) without an accumulation point. For each \( i \in \mathbb{Z}^+ \) choose recursively a path \( p^i \in P(v_0, V_i) \) with the properties that

1. \( p^{i-1} \) is a subpath of \( p^i \) when \( i \in \mathbb{Z}^+ \setminus \{0\} \)
2. The path \( p^i \) is a subpath of infinitely many of the \( x_i \).

This is clearly possible since each \( P(v_0, V_i) \) is finite.

Denote the unique infinite path containing all the \( p^i \) by \( q \). Then \( q \in X \), and \( q \) is an accumulation point of \( (x_i)_{i \in \mathbb{Z}^+} \), a contradiction. So \( X \) is compact.

Summarizing the above, we have proven that \( X \) is a Cantor set. \( \square \)

We are now able to prove a couple of lemmas.

**Lemma 2.4** Every clopen partition of \( X \) is coarser than some \( B_i \), \( i = 1, 2, \ldots \), and thus coarser than all \( B_j \), \( j \geq i \).
Proof: Let $\mathcal{P}$ be a clopen partition of $X$, thus finite. Every $M \in \mathcal{P}$ is open and therefore a union of basis elements; it is also compact so the union can be made finite. Thus there is a finite number of basis sets such that every element of $\mathcal{P}$ is a union of some of these basis sets. Let $n$ be the maximal level of these basis sets; then every basis set is a union of elements from $B_n$. Therefore $B_n$ is finer than $\mathcal{P}$.

From now we assume that an ordering $<$ is given such that $(V, E, <)$ is a simple ordered Bratteli diagram.

Lemma 2.5 Let $P_i = \bigcup_{p \in P(v_0, V_i)} \tilde{p}$, $i \in \mathbb{Z}^+$. Then $(P_i)_{i \in \mathbb{Z}^+}$ is an increasing sequence of clopen sets which covers $X \setminus \{x_{\text{min}}\}$.

Proof: Since the number of paths at each level is finite, clopenness is immediate. Given a point $y$ in $P_i$, this point must be in some $\tilde{p}$ for a nonminimal $p \in P(v_0, V_i)$. This implies that $y$ has a nonminimal edge $e$ with level $\leq i$. But then so has the unique path in $P(v_0, V_{i+1})$ containing $y$, and thus $y \in P_{i+1}$. Now pick an arbitrary point $y \in X \setminus \{x_{\text{min}}\}$. $y$ considered as an infinite path must contain a nonminimal edge $e$. Let $n$ be the level of $e$; then the unique path $p \in P(v_0, V_n)$ containing $y$ is not minimal. Thus $y \in P_n$.

We now wish to define a homeomorphism $T$ on $X$ from the simple ordered Bratteli diagram such that $(X, T)$ is a Cantor system. We progress as follows. Assume $x = (x_i)_{i=1}^{\infty} \in X$. If $x = x_{\text{max}}$ then we define $T(x) = x_{\text{min}}$. Otherwise, let $l = \min\{l \in \mathbb{Z}^+ | x_l \text{ not maximal}\}$. For $i > l$ we define $(T(x))_i = x_i$. For $i = l$ we define $(T(x))_i$ to be the edge with range $r(x_l)$ that is the successor of $x_l$. For $i < l$ assume that $(T(x))_{i+1}$ has been defined; then define $(T(x))_i$ to be the minimal edge with range equal to the source of $(T(x))_{i+1}$.

By using the canonical induced ordering on each of the sets $P(v_0, v)$, where $v$ is an edge, we note that if $p$ is a finite, nonmaximal path with source $v_0$ then $T(\tilde{p}) = (\tilde{T(p)})$ where $T(p)$ is the finite path obtained from $p$ by applying the above method in the obvious way. We also observe that given two paths $p, q \in P(v_0, v)$ for a vertex $v$, either $q = T^n(p)$ or $p = T^n(q)$ for some $n \in \mathbb{Z}^+$. (See figure 2.3)

Theorem 2.6 $(X, T)$ is a Cantor system.

Proof: We have defined the mapping $T : X \to X$. By considering instead of the simple ordered Bratteli diagram $(V, E, <)$ the simple ordered Bratteli diagram $(V, E, >)$, we obtain a mapping $T' : X \to X$; $T'$ is easily seen to be the inverse of $T$. Thus $T$ is a bijection. Let $M$ be a clopen subset of $X$ not containing $x_{\text{min}}$. By lemma 2.5, $M$ is a union of cylinder sets corresponding to nonminimal paths; The union can be taken finite and by using the definition of $T$ we note that $T^{-1}(M)$ is also a finite union of cylinder sets, thus clopen. We obtain the same for sets $M$ containing $x_{\text{min}}$ by considering complements. Thus the inverse image of a basis set is open, and $T$ is continuous. Also, $T^{-1}$ is continuous, being the corresponding mapping for the reversed diagram. We conclude that $T$ is a homeomorphism.
Figure 2.3: The paths down to a given vertex are iterated transforms of each other.

We wish to show that \((X, T)\) is minimal. Let \(O\) be any nonempty open set, and let \(\tilde{p}\) be a cylinder subset of \(O\), \(p \in P(v_0, v)\) for a vertex \(v\). We shall show that the orbit of \(\tilde{p}\) is all of \(X\). Let \(x\) be an arbitrary point in \(X\). Choose 0 = \(n_0 < n_1 < \ldots < n_j < \ldots\) to be a sequence of completely interconnected levels of \((V, E)\). Let \(n_j > \text{lev}(v)\). Let \(q_1\) be a finite subpath of \(x\) in \(P(v_0, V_{n_{i+1}})\). Let \(\tilde{q}_2\) be a superpath of \(p\) in \(P(v_0, r(q_1))\) (possible because of complete connectedness). Then we have \(q_1 = T^n(\tilde{q}_2)\) for some \(n \in \mathbb{Z}\), so \(x \in \tilde{q}_1 = T^n(\tilde{q}_2) \subset T^n(\tilde{p})\). \(\square\)

2.3 Model theorem for Cantor systems

The basis for much of our further investigations will be the following fundamental model theorem:

**Theorem 2.7** Every Cantor system is conjugate to the Cantor system generated by some simple ordered Bratteli diagram. The unique minimal path can be freely chosen to correspond to any point of the Cantor system.

This theorem is a weaker form of the results in [5, section 4], and a proof may be extracted from there, or a more detailed proof found in [4, section 3.4]. The proof depends on the construction of a sequence of “Kakutani-Rohlin” partitions of the Cantor set; the succession and inclusion properties of the sets in the partitions will allow them to be considered as finite subpaths of the corresponding simple ordered Bratteli diagram.

This theorem allows us to replace any Cantor System with a conjugate one generated by a simple ordered Bratteli diagram, and thus to work with the full structure of the latter. We shall see that this allows us to obtain several results and characterisations in a relatively simple manner.
Chapter 3

Dimension groups

3.1 Ordered Abelian groups

Definition 3.1 An ordered Abelian group is a pair $(G, G^+)$, where $G$ is an Abelian group and $G^+$ is a subset of $G$ with the following properties:

1. $G^+$ is closed under addition, i.e. $G^+ + G^+ = G^+$.
2. $G^+ - G^+ = G$.
3. $G^+ \cap (-G^+) = \{0\}$.

We will usually consider $G^+$ to be implicitly given and write the ordered group as just $G$.

Let $g_1, g_2 \in G$.

Definition 3.2 $g_1 \leq g_2 \iff g_2 - g_1 \in G^+$. Similarly for $\geq$, $<$ and $>$. Elements of $G^+$ are called positive. Elements whose negation is positive are called negative.

Definition 3.3 A positive homomorphism from an ordered Abelian group $(G_1, G_1^+)$ to an ordered Abelian group $(G_2, G_2^+)$ is a group homomorphism from $G_1$ to $G_2$ that maps elements of $G_1^+$ to elements of $G_2^+$.

Definition 3.4 Two ordered Abelian groups are isomorphic if there is a group isomorphism $\gamma$ between them such that both $\gamma$ and $\gamma^{-1}$ are positive.

The following is for our purposes an important example, as it will be used in the definition of dimension groups:

Theorem 3.1 For $m, n \in \mathbb{Z}^+$, $M$ an $n \times m$ matrix with elements in $\mathbb{Z}^+$, $(\mathbb{Z}^m, \mathbb{Z}^{+,m})$ and $(\mathbb{Z}^n, \mathbb{Z}^{+,n})$ are ordered Abelian groups. $f : \mathbb{Z}^m \to \mathbb{Z}^n$ defined by $f : u \mapsto M u$ is a positive group homomorphism. Moreover, every positive group homomorphism from $\mathbb{Z}^m$ to $\mathbb{Z}^n$ is of this form for some $n \times m$-matrix $M$ with elements in $\mathbb{Z}^+$. 
Proof: Clearly $\mathbb{Z}^+\mathbb{n}$ is closed under addition.

Let $u \in \mathbb{Z}^n$; Then $u_i = \frac{|u_i + |u_i| - |u_i| - u_i|}{2}$ for each $0 \leq i < n$ so $u \in \mathbb{Z}^+\mathbb{n}$.

If $u \in \mathbb{Z}^+\mathbb{n} \cap (\mathbb{Z}^+\mathbb{n})$ then $0 \leq u_i \leq 0$ for each $0 \leq i < n$; thus $u = 0$. So $\mathbb{Z}^n$ is an ordered Abelian group, and putting $n = m$ shows that $\mathbb{Z}^m$ is too.

$f : u \mapsto Mu$ is additive, and if $u$ is positive and $M$ has positive elements then $Mu$ is positive. So $f$ is a positive homomorphism.

Now let $f : \mathbb{Z}^m \to \mathbb{Z}^n$ be an arbitrary positive homomorphism. For each $0 \leq i < n, 0 \leq j < m$ define $M_{i,j} = (f(\delta_j))_i$, where $\delta_j$ is the basis vector of $\mathbb{Z}^m$ corresponding to $j$; Then $M\delta_j = f(\delta_j)$, and that $f : u \mapsto Mu$ for other $u \in \mathbb{Z}^m$ follows from the $\mathbb{Z}$-linearity of $f$. Since $\delta_j$ is positive so is $f(\delta_j)$ and thus $M_{i,j} = (f(\delta_j))_i \geq 0$.

Definition 3.5 Ordered Abelian groups isomorphic to $(\mathbb{Z}^n, \mathbb{Z}^+\mathbb{n})$ for some $n \in \mathbb{Z}^+$ will be denoted as standard.

3.2 Definition of dimension groups

Definition 3.6 A directed sequence of ordered Abelian groups is a sequence $(G_n)_{n \in \mathbb{Z}^+}$ of ordered Abelian groups, and for each pair of $m, n \in \mathbb{Z}^+, m < n$ a positive homomorphism $\gamma^m_n : G_m \to G_n$; such that whenever $l, m, n \in \mathbb{Z}^+, l < m < n$, the following diagram commutes:

Definition 3.7 A direct limit of a directed sequence $((G_n)_{n \in \mathbb{Z}^+}, (\gamma^m_n)_{m,n \in \mathbb{Z}^+, m < n})$ of ordered Abelian groups is an ordered Abelian group $G$ together with a sequence $\gamma_n : G_n \to G$ such that the following diagram commutes for all $m, n \in \mathbb{Z}^+, m < n$:

and solving the following universal problem:

For every ordered Abelian group $H$ and sequence $(\eta_n)$ of positive homomorphisms $\eta_n : G_n \to H$ such that whenever $m, n \in \mathbb{Z}^+, m < n$, the following diagram commutes:
there is a unique positive homomorphism \( \eta: G \to H \) such that whenever \( n \in \mathbb{Z}^+ \), the following diagram commutes:

\[
\begin{array}{ccc}
G_n & \xrightarrow{\gamma_m} & H \\
\downarrow{\eta_m} & & \downarrow{\eta} \\
G & \xrightarrow{\eta} & H
\end{array}
\]

**Theorem 3.2** Every directed sequence of ordered Abelian groups has a direct limit, and any two such limits are order isomorphic by an isomorphism intertwining the \( \gamma^n \)-mappings.

**Proof:** Assume \( G_i \) with mappings \( \gamma^m_n \) is a directed sequence.

Consider the set of sequences \( (g_i)_{i \in \mathbb{Z}^+} \) where \( g_i \in G_i \), \( i \in \mathbb{Z}^+ \) and such that there is an \( N \) dependent of \( (g_i) \) with \( g_n = \gamma^m_n(g_m) \) for all integers \( n > m \geq N \). Take as \( G \) the quotient group of this Abelian group with the group of such sequences that are eventually 0. We define \( G^+ \) as the classes containing sequences that are eventually positive.

\( G^+ \) is clearly closed under addition, and each sequence in \( G \) is the difference of its positive and negative parts. If a sequence is eventually both negative and positive it is eventually 0. So \( G \) is an ordered Abelian group.

We let \( \gamma_n(g) = [h] \) where \( h_i = \gamma^i_n(g) \), \( i > n \) and \( h_i = 0 \) otherwise. Then \( \gamma_n \) is well defined, and each \( \gamma_n \) is a positive homomorphism.

The commutation of the first diagram of definition 3.7 follows from the fact that changing a finite number of terms in \( h \) does not change \([h]\].

Given \( H \) and \((\eta_i)\) we must have \( \eta((g_i)) = \lim_{i \to \infty} \eta_i(g_i) \) since the limit sequence is eventually constant. This works and uniquely defines \( \eta \).

Thus \( (G, (\gamma_i)) \) is a direct limit of \((G_i, (\gamma^m_n))\).

Assume that a sequence has the direct limits \( (G, (\gamma_i)) \) and \( (G', (\gamma'_i)) \). \((G, (\gamma_i)) \) fulfils the premises for the universal property of \((G', (\gamma'_i))\), and the \( \eta \) so constructed is a positive homomorphism from \( G' \) to \( G \); Passing the other way we obtain the inverse of \( \eta \), so the limits are isomorphic; the isomorphisms also must intertwine the mappings, as shown on this diagram.
We are now ready to define dimension groups:

**Definition 3.8** A **dimension group** is an ordered Abelian group that is the direct limit of a directed sequence containing only standard ordered Abelian groups.

The following connects dimension groups with Bratteli diagrams:

**Definition 3.9** The **associated dimension group** of a Bratteli diagram is the direct limit of the following directed sequence:

Given \( n \in \mathbb{Z}^+ \), let \( G_n = \mathbb{Z}^{V_n} \) and \( G_n^+ = \mathbb{Z}^{+ V_n} \); Given \( m, n \in \mathbb{Z}^+ \), \( m < n \), \( g \in G_m \) let

\[
\gamma^m_n (g) = (\prod_{i=m+1}^{n} M_i) g.
\]

**Definition 3.10** A dimension group that arises from a simple Bratteli diagram is said to be simple.

An **order ideal** of \( G \) is a subset \( I \) of \( G \) such that \( (I, I \cap G^+) \) is an ordered subgroup of \( G \) that is hereditary, i.e. for all \( a \in G, g \in I \cap G^+ \) such that \( 0 \leq a \leq g \), we have \( a \in I \cap G^+ \). **Trivial** order ideals of \( G \) are 0 and \( G \).

By a theorem in [1] a dimension group \( G \) is simple if and only if it has no nontrivial order ideal.
Chapter 4

The simple dimension group $K^0(X, T)$

4.1 Definition of $K^0(X, T)$

Let $(X, T)$ be a Cantor system.

If $M$ is a subset of $X$, we denote the characteristic function of $M$ (the function being 1 on $M$ and 0 on $X \setminus M$) by $\chi_M$.

**Theorem 4.1** For $\chi_M$ to be continuous it is necessary and sufficient that $M$ is clopen.

**Proof:** $M = \chi_M^{-1}(\{1\})$ so clopenness is necessary. $\chi_M(X) \subset \{0, 1\}$, so $\emptyset, \{0\}, \{1\}$, and $\{0, 1\}$ are the only open subsets of the range. Thus it is sufficient that $X \setminus M = \chi_M^{-1}(\{0\})$ and $M$ are open, i.e. $M$ must be clopen.

By $C(X, Z)$ we mean the set of continuous functions from $X$ to $Z$. This set is an Abelian group under pointwise addition.

$B(X, T) = \{g - g \circ T | g \in C(X, Z)\}$ is a subgroup of it. Thus we can create the quotient group $K^0(X, T) = C(X, Z)/B(X, T)$. We denote the equivalence class in the quotient of a function $f \in C(X, Z)$ by $[f]$.

We define $K^0+(X, T) = \{[f] | f \geq 0, f \in C(X, Z)\}$. Since $[f] + [g] = [f + g], K^0+(X, T)$ is closed under addition. Also, $[f] = [f^+] - [f^-]$ where $f^+$ and $f^-$ are the positive and negative parts of $f$, respectively. To show that $K^0+(X, T) \cap (-K^0+(X, T)) = \{[0]\}$ we need some more machinery; it will be proved in corollary 4.7.

The main aim of this chapter will be to prove that $K^0(X, T)$ is a dimension group isomorphic to those associated to simple ordered Bratteli diagrams generating Cantor systems conjugate to $(X, T)$. Clearly, this fact is not changed by passing to a conjugate system, and so in the following, we will assume that $(X, T)$ is realized on a simple ordered Bratteli diagram$(V, E, <, r, s)$ with the notation from section 2.2.

On the way to the proof, we will establish some useful theorems allowing us to examine $K^0(X, T)$ by working on a corresponding simple ordered Bratteli diagram.

Consider any $f \in C(X, Z)$. The set $f(X)$ is a compact subset of $Z$, thus finite. $f$ defines a clopen partition $\{f^{-1}(z) | z \in Z\}$ of $X$. 

18
Definition 4.1 \( \text{lev}(f) \) is the least natural number such that \( B_{\text{lev}(f)} \) is a finer partition than \( \{ f^{-1}(z) \mid z \in \mathbb{Z} \} \).

By lemma 2.4 this is well defined. Then \( f \) is constant on each element of \( B_n, n \geq \text{lev}(f) \), which means that \( f = \sum_{p \in P(v_0,v_n)} f(\tilde{p}) \chi_{\tilde{p}} \), where \( f(\tilde{p}) \) is the constant value of \( f \) on \( \tilde{p} \).

Definition 4.2 For \( v \in V_n, n \geq \text{lev}(f) \) we then define

\[
  f_v = \sum_{p \in P(v_0,v)} f(\tilde{p})
\]

and

\[
  \vec{f}_n = (f_v)_{v \in V_n}
\]

The following theorem will later be used to write out explicitly certain positive group homomorphisms.

Theorem 4.2 We have \( \vec{f}_{n+1} = M^{n+1} \vec{f}_n, n \geq \text{lev}(f) \), where \( M^{n+1} \) is the incidence matrix from level \( n \) to \( n + 1 \); \( f \mapsto f_v \) and \( f \mapsto \vec{f}_n \) are additive mappings on their domains of definition for each \( v \in V_n \) and \( n \in \mathbb{Z}^+ \).

Proof: Additivity is obvious.

\[
\begin{align*}
  \vec{f}_{n+1} &= \left( \sum_{p \in P(v_0,v)} f(\tilde{p}) \right)_{v \in V_{n+1}} \\
  &= \left( \sum_{u \in V_n} \sum_{e \in P(u,v)} \sum_{q \in P(v_0,u)} f(\tilde{q}) \right)_{v \in V_{n+1}} \\
  &= \left( \sum_{u \in V_n} \sum_{e \in P(u,v)} f_u \right)_{v \in V_{n+1}} = \left( \sum_{u \in V_n} M^{n+1, v, u} f_u \right)_{v \in V_{n+1}} \\
  &= M^{n+1} \vec{f}_n
\end{align*}
\]

We will have use for the following lemma in constructing functions with particular level vectors:

Lemma 4.3 For any \( n \in \mathbb{Z}^+ \) and any function \( \vec{u} : V_n \rightarrow \mathbb{Z} \) there exists an \( f \in C(X, \mathbb{Z}) \) such that \( \text{lev}f \leq n, \vec{f}_n = \vec{u} \) and \( f(X) \subset u(V_n) \cup \{0\} \).

Proof: For each \( x \in X \) there are unique \( v \in V_n, p \in P(v_0, v) \) such that \( x \in \tilde{p} \). Define \( f(x) = \vec{u}(v) \), whenever \( p \) is minimal; and \( f(x) = 0 \) otherwise.
Figure 4.1: How to find a $g$ such that $f = g - g \circ T$

4.2 Equivalence and comparision criteria for the functions in $C(X, Z)$

Theorem 4.4 $f = g - g \circ T$ for some $g \in C(X, Z)$ if and only if there is some $n \geq \text{lev}(f)$ such that $\vec{f}_n = 0$.

Proof:

(†) Assume there is some $n \geq \text{lev}(f)$ such that $\vec{f}_n = 0$. Thus we have $\sum_{p \in P(v_0, v)} f(\tilde{p}) = 0 \quad \forall v \in V_n$. Define $g \in C(X, Z)$ as in figure 4.1 by

$$g(\tilde{p}) = \sum_{q \in P(v_0, r(p))} f(\tilde{q}), \; p \in P(v_0, V_n).$$

If $p$ is a minimal path in $P(v_0, V_n)$, $g(\tilde{p}) = \sum_{q \in P(v_0, r(p))} f(\tilde{q}) = 0$; thus $(g \circ T)(\tilde{p}) = \sum_{q \in P(v_0, r(p))} f(\tilde{q}), \; p \in P(v_0, V_n)$; and $f = g - g \circ T$.

(‡) Assume $f = g - g \circ T$, $g \in C(X, Z)$. Define $M = X \setminus g^{-1}(g(x_{min}))$; $M$ is then clopen and does not contain $x_{min}$. By lemma 2.5 there must then be some $i \in \mathbb{Z}^+ \setminus \{0\}$ such that $M \subset \bigcup_{p \in P(v_0, V_n)} \tilde{p}$. Then for $n \geq \max(i, \text{lev}(f))$ we may write $g - g(x_{min})1 = \sum_{p \in P(v_0, V_n)} b_p \chi_{\tilde{p}}$. If $p \in P(v_0, V_n)$ and $b_p \neq 0$, $\tilde{p} \subset M \subset \bigcup_{p \in P(v_0, V_n)} \tilde{p}$ and so $p$ is not minimal. We then have the situation in figure 4.2. Furthermore
Figure 4.2: For each vertex \( v \), \( g - g \circ T \) sums to 0 on \( P(v_0, v) \)

\[
g - g(x_{min}) = \sum_{p \in P(v_0, V_n)} b_p \chi_{\hat{p}} \quad \text{and} \quad f = g - g \circ T = g - g(x_{min}) + g(x_{min}) \circ T
\]

\[
= \sum_{p \in P(v_0, V_n)} b_p (\chi_{\hat{p}} - \chi_{\hat{p} \circ T})
\]

\[
= \sum_{p \in P(v_0, V_n)} b_p (\chi_{\hat{p}} - \chi_{\text{pred}(p)}).
\]

Now for \( v \in V_n \):

\[
f_v = \sum_{p \in P(v_0, v)} f(\hat{p}) = \sum_{p \in P(v_0, v)} \sum_{q \in P(v_0, V_n)} b_q (\chi_q(\hat{p}) - \chi_{\text{pred}(q)}(\hat{p}))
\]

\[
= \sum_{p \in P(v_0, v)} b_p - \sum_{p \in P(v_0, v)} b_{\text{succ}(p)} = 0.
\]

\[\square\]

**Theorem 4.5** \([f] = [g], f, g \in C(X, Z) \) if and only if there is some \( n \geq \max(\text{lev}(f), \text{lev}(g)) \) such that \( \tilde{f}_n = \tilde{g}_n \).

**Proof:** By definition, \([f] = [g] \) if and only if \( f - g = h - h \circ T \) for some \( h \in C(X, Z) \). This is by theorem 4.4 equivalent to the existence of an \( n \geq \text{lev}(f - g) \) such that \( (f - g)_v = 0 \) for all \( v \in V_n \).

(\( \uparrow \)) We have \( \text{lev}(f - g) \leq \max(\text{lev}(f), \text{lev}(g)) \). If \( f_v = g_v \) for all \( v \in V_n, n \in \mathbb{Z}^+, n \geq \max(\text{lev}(f), \text{lev}(g)) \), we obviously have \( (f - g)_v = 0 \) by linearity.
(\$) Let $n' = \max(n, \text{lev}(f), \text{lev}(g))$. If $(f - g)_u = 0$ for all $u \in V_n$, we have for $v \in V_{n'}$ ($M^i$ are the incidence matrices):

$$
(f_v)_{v \in V_{n'}} - (g_v)_{v \in V_{n'}} = ((f - g)_v)_{v \in V_{n'}} = (\prod_{i=n}^{n'-1} M^i)((f - g)_u)_{u \in V_n}
$$

$$
= (\prod_{i=n}^{n'-1} M^i)(0)_{u \in V_n} = (0)_{v \in V_{n'}}
$$

so $f_v - g_v = 0$ for all $v \in V_{n'}$.

\[ \Box \]

**Theorem 4.6** $[f] \in K^{0+}(X, T)$, $f \in C(X, \mathbb{Z})$ if and only if there is some $n \geq \text{lev}(f)$ such that $f_v \geq 0$ for all $v \in V_n$.

**Proof:** By definition, $[f] \in K^{0+}(X, T)$ if and only if $[f] = [g]$ for some $g \in C(X, \mathbb{Z})$, $g \geq 0$.

(\$\$) Construct $g$ according to lemma 4.3 with $\vec{u} = \vec{f}_n$.

(\$\$) Obviously, if $g \geq 0$, $g_v \geq 0$ for all $v \in V_n$, $n \geq \text{lev}(g)$. By theorem 4.5 there is an $n \geq \max(\text{lev}(f), \text{lev}(g))$ such that $\vec{f}_v = \vec{g}_v \geq 0$ for all $v \in V_n$.

\[ \Box \]

**Corollary 4.7** $K^{0}(X, T)$ is an ordered Abelian group.

**Proof:** We have left to show that $K^{0+}(X, T) \cap (-K^{0+}(X, T)) = \{0\}$. Let $[f] \in K^{0+}(X, T) \cap (-K^{0+}(X, T))$; then there must be $m, n \in \mathbb{Z}^+$ such that $\vec{f}_m \geq 0$ and $-\vec{f}_n \geq 0$; then $\vec{f}_{\max(m, n)} = 0$ and $[f] = [0]$.

\[ \Box \]

### 4.3 $K^{0}(X, T)$ is a simple dimension group

We now wish to show that we can define the mappings to make $K^{0}(X, T)$ into the dimension group associated to the Bratteli diagram $(V, E)$. Recall definitions 3.9 and 3.10.

**Definition 4.3** For each $n \in \mathbb{Z}^+$, we define the function $\gamma_n : \mathbb{Z}^{V_n} \to K^{0}(X, T)$ as follows. Given $f \in C(X, \mathbb{Z})$, $\text{lev}(f) \leq n$: $\gamma_n(\vec{f}_n) = [f]$.

**Theorem 4.8** For each $n \in \mathbb{Z}^+$, $\gamma_n$ is a positive homomorphism.

**Proof:** By lemma 4.3 every element of $\mathbb{Z}^{V_n}$ is of the form $\vec{f}_n$ for some $f \in C(X, \mathbb{Z})$, $\text{lev}(f) \leq n$, and that $\gamma_n$ is well defined then follows from theorem 4.5. By theorem 4.2, $\gamma_n(\vec{f}_n + \vec{g}_n) = \gamma_n(\vec{f} + \vec{g}_n) = [f + g] = [f] + [g]$, so $\gamma_n$ is linear. Let $\vec{f}_n$ be positive; by theorem 4.6, $[f] \geq 0$. 

\[ \Box \]
**Theorem 4.9** The following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{Z}^m & \xrightarrow{\gamma_m} & K^0(X, T) \\
\gamma_n & \xrightarrow{\gamma_n} & \\
\mathbb{Z}^n & \xrightarrow{\gamma_n} & \\
\end{array}
\]

*Proof:* Let \(\tilde{f}_m, f \in C(X, \mathbb{Z}), \text{lev}(f) \leq m\) be an arbitrary element of \(\mathbb{Z}^m\); then \(\gamma_n(\gamma_m(\tilde{f}_m)) = \gamma_n(f \circ_m \tilde{f}_m) = [f] = \gamma_m(\tilde{f}_m)\). \hfill \Box

**Theorem 4.10** \(K^0(X, T)\) with the mappings \(\gamma_n\) is the simple dimension group associated with the Bratteli diagram \((V, E)\).

*Proof:* Let \(H\) be an ordered Abelian group, and let \(\eta_n : \mathbb{Z}^n \to H, n \in \mathbb{Z}^+\) be positive homomorphisms such that the following diagram commutes for each \(m, n \in \mathbb{Z}^+, m < n:\)

\[
\begin{array}{ccc}
\mathbb{Z}^m & \xrightarrow{\eta_m} & H \\
\gamma_n & \xrightarrow{\gamma_n} & \\
\mathbb{Z}^n & \xrightarrow{\eta_n} & \\
\end{array}
\]

Then in order for \(\eta : K^0(X, T) \to H\) to be a positive homomorphism such that the diagram

\[
\begin{array}{ccc}
K^0(X, T) & \xrightarrow{\eta} & H \\
\gamma_n & \xrightarrow{} & \\
\mathbb{Z}^n & \xrightarrow{} & \\
\end{array}
\]

commutes for each \(n \in \mathbb{Z}^+\), we must have \(\eta([f]) = \eta(\gamma_n(f \circ_n \tilde{f}_n)) = \eta_n(f \circ_n \tilde{f}_n)\), so if \(\eta\) exists it is unique.

To see that \(\eta\) is well defined, assume first that \(f \in C(X, \mathbb{Z}), n \in \mathbb{Z}^+, n > \text{lev}(f)\); then \(\eta_n(f \circ_n \tilde{f}_n) = \eta_n(f \circ_n \gamma_n(\text{lev}(f)) \circ_n \tilde{f}_n) = \eta_n(f \circ_n \gamma_n(\text{lev}(f)) \circ_n \tilde{f}_n)\); so the expression is independent of \(n\);

Assume then that \([f] = [g], f, g \in C(X, \mathbb{Z})\); then by theorem 4.5 there exists an \(n \in \mathbb{Z}^+, n \geq \max(\text{lev}(f), \text{lev}(g))\) such that \(f \circ_n \tilde{f}_n = g_n \circ_n \tilde{g}_n\); so the value of the expression is independent of the choice of representative. Thus \(\eta([f])\) is well defined.

We then need to prove that \(\eta\) is a positive homomorphism. \(\eta([f] + [g]) = \eta(f \circ_n + g_n) = \eta(f \circ_n + g_n) = \eta(f) + \eta(g)\) for \(n \geq \text{lev}(f), \text{lev}(g)\). If \([f]\) is positive we may assume \(f\) positive and \(\eta([f]) = \eta(f \circ_n \gamma_n(\text{lev}(f))) \geq 0\). \hfill \Box

23
4.4 Characteristic functions

Assume \((X, T)\) is a Cantor system.

If \(f \in C(X, Z)\) is equivalent to a characteristic function it is clearly necessary that \([0] \leq [f] \leq [1]\). The following theorem shows the inverse.

**Theorem 4.11** If \(f \in C(X, Z)\), where \((X, T)\) is a Cantor system, and \([0] \leq [f] \leq [1]\); then \([f] = [\chi_M]\) for some clopen set \(M \subset X\).

**Proof:** By theorem 4.6 there exist \(n_1, n_2 \in \mathbb{Z}^+\) such that \(f_v \geq 0\) for all \(v \in V_{n_1}\) and \(1_v - f_v \geq 0\) for all \(v \in V_{n_2}\). We may then pass to the common level \(n = \max(n_1, n_2)\). We have \(1_v = |P(v_0, v)|\) so we get \(0 \leq f_v \leq |P(v_0, v)|\). We may then define \(g \in C(X, Z)\) as 1 on the cylinder sets of the \(f_v\) first paths down to a given vertex \(v \in V_n\), and 0 on cylinder sets of the \(|P(v_0, v)| - f_v\) last paths. Then we get the situation in figure 4.3; clearly \(g\) is a characteristic function, and \(g_v = f_v\) for each \(v \in V_n\); so \([f] = [g]\). \(\square\)

4.5 The full group

**Definition 4.4** By the **full group** on \((X, T)\) we mean the group of homeomorphisms \(g : X \to X\) with the following property: There exists a clopen partition \(\mathcal{P}\) of \(X\) such that for each element \(M \in \mathcal{P}\), \(g|_M = T^n|_M\) for some \(n \in \mathbb{Z}\) dependent on \(M\).

**Theorem 4.12** If \(\chi_A - \chi_B = g - g \circ T\), where \(A\) and \(B\) are clopen subsets of \(X\) and \(g \in C(X, Z)\), then there is a homeomorphism \(h : X \to X\) in the full group on \((X, T)\) such that \(h^2 = \text{Id}_X\), \(h(A) = B\), \(h(B) = A\), and \(h|_{X \setminus A \setminus B} = \text{Id}_{X \setminus A \setminus B}\).
Proof: By theorem 4.5, there is an $n \geq \max(\text{lev}(\chi_A), \text{lev}(\chi_B))$ such that $(\chi_A)_v = (\chi_B)_v \forall v \in V_n$. Let $P = B_n$.

$\chi_A - \chi_B$ can only take the values $-1, 0, 1$ in a point. For each vertex $v \in V_n$, we consider the sets $M^i_v = \{p \in P(v_0, v)|\chi_A(\tilde{p}) - \chi_B(\tilde{p}) = i\}$, $i = -1, 0, 1$. Clearly $|M^i_v| = |M^{-i}_v|$, $i = -1, 0, 1$ since $(\chi_A)_v = (\chi_B)_v$. We thus have the situation in figure 4.4, and wish to define $h$ as shown. Define $n_v$ as the unique order isomorphism from $P(v_0, v)$ to $\{1, \ldots, |P(v_0, v)|\}$. Similarly, define $n^i_v$ as the order isomorphism from $M^i_v$ to $\{1, \ldots, |M^i_v|\}$. Define $h$ by

$$h(x) = T^{n_v^{-1}(n^i_v(p)) - n_v(p)}(x), \quad \text{for } x \in \tilde{p}, p \in M^i_v, i = -1, 0, 1, v \in V_n.$$ 

Since $\{M^i_v| i = -1, 0, 1, v \in V_n\}$ is a partition of $P(v_0, V_n)$ and $\{\tilde{p}|p \in P(v_0, V_n)\}$ is a partition of $X$, $h$ is well defined on $X$. Moreover, for each $p$, $h|_{\tilde{p}} = T^i|_{\tilde{p}}$ for some $i$ dependent on $p$, such that $P = B_n$ is a partition of $X$ with the required property. Clearly, $h^2 = \text{Id}_X$, $h(A) = B$, $h(B) = A$, and $h|_{X\setminus A \setminus B} = \text{Id}_{X\setminus A \setminus B}$. Noting that $h|_{\tilde{p}}$ is continuous for each $p$ and that each $\tilde{p}$ is open, we see that $h$ is continuous at each point of $X$. Since closed (compact) sets are mapped to closed sets, the inverse of $h$ is also continuous. $\square$
Chapter 5

Measures and integration on Cantor systems

5.1 Measures on Cantor sets

We will recall some facts about measures, algebras and $\sigma$-algebras.

Let $X$ be a Cantor set.

By an algebra on $X$ we mean a family of subsets of $X$ which is closed under the set operations of finite intersection, finite union, and complement. The family $C$ of clopen sets of $X$ is an algebra.

By a $\sigma$-algebra on $X$ we mean a family of subsets of $X$ which is closed under the set operations of countable intersection, countable union, and complement. Any $\sigma$-algebra is an algebra. The family $\mathcal{P}(X)$ of all subsets of $X$ is a $\sigma$-algebra, and the intersection of any family of $\sigma$-algebras is again a $\sigma$-algebra. Thus the following definition makes sense:

The Borel algebra $\mathcal{B}(X)$ of $X$ is the least $\sigma$-algebra on $X$ containing the open sets of $X$.

Since $X$ is a Cantor set, every open set is the countable union of clopen sets, and so we could equally use "clopen" instead of "open" in this definition.

A Borel set is any member of $\mathcal{B}(X)$.

A (finite) measure is a function $\mu : \mathcal{A} \rightarrow \mathbb{R}^+$ on a $\sigma$-algebra $\mathcal{A}$ with the property: $\mu(\bigcup_{i=1}^{\infty} M_i) = \sum_{i=1}^{\infty} \mu(M_i)$ for every pairwise disjoint sequence of elements $(M_i)_{i=1}^{\infty}$ of $\mathcal{A}$.

A measure on the Borel algebra of $X$ is known as a Borel measure on $X$.

If in addition $\mu(X) = 1$, we say that $\mu$ is a probability measure on $X$.

We have the following general theorem for measures on arbitrary sets:

**Theorem 5.1** Assume $\mu : \mathcal{A} \rightarrow \mathbb{R}^+$ is an additive set function on an algebra $\mathcal{A}$ in a set $X$, such that whenever $(A_i)$ is a sequence of pairwise disjoint elements of $\mathcal{A}$ such that $\bigcup_{i=0}^{\infty} A_i \subset \mathcal{A}$, then $\mu(\bigcup_{i=0}^{\infty} A_i) = \sum_{i=0}^{\infty} \mu(A_i)$.

Then $\mu$ has a unique extension to a measure on the least $\sigma$-algebra containing $\mathcal{A}$.

This is slightly weaker than [2, proposition 1.13], and a proof can be found there.

This theorem implies the following.
Theorem 5.2  Every additive set function \( \mu : C \to \mathbb{R}^+ \) can be uniquely extended to a Borel measure.

Proof: If \( (A_i) \) is a sequence of pairwise disjoint elements of \( C \) such that \( \bigcup_{i=0}^{\infty} A_i \subset C \), then all except a finite number of the \( A_i \) are empty, so it follows that \( \mu(\bigcup_{i=0}^{\infty} A_i) = \sum_{i=0}^{\infty} \mu(A_i) \). Since \( \mathcal{B}(X) \) is the least \( \sigma \)-algebra containing \( C \), the result follows. \( \square \)

5.2  Invariant measures on Cantor systems

Let \( (X, T) \) be a Cantor system. \( T \) is a homeomorphism and homeomorphisms map Borel sets to Borel sets. This can be seen by noting that the image of \( C \) is \( C \) and that a homeomorphism preserves unions, intersections and complements, being a bijection.

A Borel measure \( \mu \) on \( X \) is \( T \)-invariant if for every Borel set \( M \) we have \( \mu(M) = \mu(T^{-1}(M)) \). It is clear that this holds if it holds for clopen sets, by uniqueness of extension.

Definition 5.1  By \( M(X, T) \) we mean the set of \( T \)-invariant Borel measures on \( X \).

We observe that since the orbit of an open set is the entire of \( X \), every open set must have non-zero measure unless the measure is identically 0.

Assume now that the Cantor system is generated by a simple ordered Bratteli diagram \( (V, E, <) \). We shall show how each \( T \)-invariant measure corresponds to a sequence of vectors derived from the Bratteli diagram.

First we note that in order to identify a Borel measure, it is enough to know its value on cylinder sets. This is because every clopen set is a (finite) disjoint union of cylinder sets.

Secondly, we note that because the cylinder sets corresponding to paths to a common vertex are iterated \( T \)-transforms of each other, all cylinder sets corresponding to a given range vertex must have the same measure. Thus the following is well defined (see figure 5.1):

Definition 5.2  Let \( \mu \in M(X, T) \). We then define

\[
\mu_v = \mu(\tilde{p}), p \in P(v_0, v)
\]

and

\[
\vec{\mu}_n = (\mu_v)_{v \in V_n}
\]

Theorem 5.3  If \( \mu \in M(X, T) \), and \( M^{n+1} \) is the incidence matrix from level \( n \) to \( n+1 \) of the simple ordered Bratteli diagram \( (V, E, <) \), then:

1. \( \mu_{v_0} = \mu(X) \)

2. \( \mu_v \in \mathbb{R}^+ \) for each \( v \in V \)

3. \( \vec{\mu}_n = (M^{n+1})^T \vec{\mu}_{n+1} \) for \( n \in \mathbb{Z}^+ \).
Figure 5.1: Measure vectors

Proof:

1. \( \mu_{v_0} = \mu(\bar{p}) = \mu(X) \) where \( p \) is the zero length path.

2. \( \mu_v = \mu(\bar{p}) \geq 0, p \in P(v_0, v) \).

3. Let \( p_v \) be a path in \( P(v_0, v) \) for each vertex \( v \).

\[
\vec{\mu}_n = (\mu(\bar{p}_v))_{v \in V_n} = \left( \sum_{u \in V_{n+1}} \sum_{e \in P(u, v)} \mu(\bar{p}_u) \right)_{v \in V_n} = \left( \sum_{u \in V_{n+1}} M^{n+1}_{u,v} \mu_u \right)_{v \in V_n} = (M^{n+1})^T \vec{\mu}_{n+1}.
\]

We see that each \( T \)-invariant Borel measure \( \mu \) on \( X \) gives rise to a sequence of vectors with the properties in theorem 5.3. The next theorem shows how any such sequence of vectors arises from a unique measure.

**Theorem 5.4** Let \((a_n)_{n \in \mathbb{Z}^+}\) be a sequence of vectors \( a_n \in \mathbb{R}^{+V_n} \) such that for all \( n \in \mathbb{Z}^+ \)

\[
a_n = (M^{n+1})^T \alpha_{n+1}.
\]

Then there is a unique measure \( \mu \in M(X, T) \) on \( X \) such that \( a_n = \vec{\mu}_n \) for each \( n \in \mathbb{Z}^+ \).
Proof: We will first define $\mu$ on cylinder sets, and then on clopen sets, showing that it is additive on $\mathcal{C}$. Theorem 5.2 then immediately extends $\mu$ to a Borel measure, which we will show is $T$-invariant.

First we wish to define $\mu(p) = (a_n)_v$ for $p \in P(v_0, v), v \in V_n$. If $p_1 = p_2$ for $p_1 \in P(v_0, v_1), p_2 \in P(v_0, v_2), \text{lev}(v_1) = i_1 \leq \text{lev}(v_2) = i_2$ then we must have that $P(v_1, V_{i_2}) = P(v_1, v_2)$ is a one-point set and so

$$(a_i)_v = \left( \prod_{n=i_{i_2}+1}^{i_2} M^n \right) (a_{i_2})_v = \sum_{v \in V_{i_2}} |P(v_1, v)| (a_{i_2})_v = |P(v_1, v_2)| (a_{i_2})_v = (a_{i_2})_v.$$ 

So this is well defined.

If each of $O_1$ and $O_2$ is a finite family of disjoint cylinder sets, and $\cup O_1 = \cup O_2 = M$, then we wish to show that $\sum_{b \in O_2} \mu(b) = \sum_{b \in O_2} \mu(b)$.

First, since the intersection of two cylinder sets is either empty or equal to one of the sets, the common refinement of $O_1$ and $O_2$ as partitions of $M$ is itself a finite family $O_3$ of disjoint cylinder sets with $\cup O_3 = M$. By considering first $O_1$ and $O_3$ and then $O_2$ and $O_3$, we reduce the problem to the case where $O_2$ is a refinement of $O_1$.

In this case, each element $b \in O_1$ is a finite disjoint union of elements from $O_2$, and so we may reduce further to the case where $O_1$ is a one set family, i.e. $M$ is a cylinder set.

To show this case, we will now use induction on the cardinality of $O_2$. Assume proven for $|O_2| < n$, and consider the case $|O_2| = n$.

Consider first the case where $O_2$ contains a proper subfamily $O_3$ whose union $N$ is a cylinder set. Then we reduce the problem to the cases $M, (O_2 \setminus O_3) \cup \{N\}$ and $N, O_3$, which follow by the induction statement.

Otherwise, we have either that $O_2 = \{M\} = O_1$, in which case we are finished, or we have $M = \hat{p}, p \in P(v_0, v), v \in V_l, l \in \mathbb{Z}^+$ and $O_2 = \{\hat{q}|q \in P(v_0, V_{l+1}), p \text{ subpath of } q\}$. But then $\sum_{b \in O_2} \mu(b) = \mu(M) = (a_l)_v$ and

$$\sum_{b \in O_2} \mu(b) = \sum_{q \in P(v_0, V_{l+1}), p \text{ subpath of } q} (a_{l+1})_{r(q)} = \sum_{e \in P(v, V_{l+1})} \mu(1)_{r(e)} = \sum_{w \in V_{l+1}} |P(v, w)| (a_{l+1})_w = \sum_{w \in V_{l+1}} M^{l+1}_{w,v} (a_{l+1})_w = ((M^{l+1})^T a_{l+1})_v = (a_l)_v.$$ 

We are now able to define $\mu$ on clopen sets. Let $M$ be a clopen set, then $M = \cup O$ for some finite disjoint family of cylinder sets; define $\mu(M) = \sum_{b \in O} \mu(b)$; by the above this is well defined.

To show additivity, let $M_1 = \cup O_1$ and $M_2 = \cup O_2$ be disjoint clopen sets; then

$$\mu(M_1 \cup M_2) = \mu(\cup(O_1 \cup O_2)) = \sum_{b \in O_1 \cup O_2} \mu(b) = \sum_{b \in O_1} \mu(b) + \sum_{b \in O_2} \mu(b) = \mu(O_1) + \mu(O_2).$$

Thus by theorem 5.2 $\mu$ can be uniquely extended to a Borel measure on $X$. 29
We now must show that \( \mu \) is \( T \)-invariant. First, when \( p \) is a nonminimal path, \( p \in P(v_0, v), v \in V_i, l \in \mathbb{Z}^+ \), \( \mu(\tilde{p}) = (a_l)_v = \mu(T^{-1}(\tilde{p})) \). If \( M \) is a clopen set not containing \( x_{\text{min}} \), then by lemma 2.5 there is a family \( O \) of cylinder sets of nonminimal paths covering \( M \). Let \( n \) be the maximal level of the sets in \( O \). Let \( O' \) be a partition of \( M \) into cylinder sets all of level \( \geq n \). Then \( O' \) must contain only cylinder sets of nonminimal paths. Thus \( \mu(M) = \mu(T^{-1}(M)) \).

By taking complements we get \( T \)-invariance of \( \mu \) on clopen sets containing \( x_{\text{min}} \). This means that the Borel measures \( \mu \) and \( \mu \circ T^{-1} \) agree on clopen sets; thus \( \mu \circ T^{-1} \) is an extension of \( \mu|_C \) to a Borel measure; since extension is unique, \( \mu \circ T^{-1} = \mu \).

Clearly, \( \mu_v = \mu(\tilde{p}) = (a_l)_v \) for \( p \in P(v_0, v), v \in V_i, l \in \mathbb{Z}^+ \), and so \( \tilde{\mu}_n = a_n \) for each \( n \in \mathbb{Z}^+ \).

The next theorems give us an indication of what vectors are the level vectors of measures.

**Theorem 5.5** Let \( n \in \mathbb{Z}^+ \). For an element \( a \in \mathbb{R}^{+V_n} \), the following are equivalent:

1. There is a measure \( \mu \in M(X, T) \) such that \( a = \tilde{\mu}_n \)
2. For each \( m \in \mathbb{Z}^+, m \geq n \), \( a = \prod_{i=n+1}^{m}(M^i)^T b \) for some \( b \in \mathbb{R}^{+V_m} \)
3. \( a \in \bigcap_{m=n}^{\infty}\{b \in \mathbb{R}^{+V_n}\mid b = \prod_{i=n+1}^{m}(M^i)^T c, c \in \mathbb{R}^{+V_m}\} \).

**Proof:** (1) \( \Rightarrow \) (2) follows by setting \( b = \tilde{\mu}_m \). (3) is just a reformulation of (2).

To show (2) \( \Rightarrow \) (1), we will use theorem 5.4.

By Zorn’s lemma define \((a_m)\) to be a sequence (with any positive index interval \( I \)) such that

1. \( a_n = a \)
2. \( a_m = (M^{m+1})^T a_{m+1} \) when \( m, m+1 \in I \).
3. For each \( j \in I, m \in \mathbb{Z}^+, m \geq j \), \( a_j = \prod_{i=j+1}^{m}(M^i)^T b \) for some \( b \in \mathbb{R}^{+V_m} \)
4. \((a_m)\) is not a proper subsequence of any sequence with these properties.

First, the lower index bound must be 0, as otherwise we could extend \((a_m)\) by setting \( a_m = \prod_{l=m+1}^{n}(M^l)^T a \) for \( m \leq n \).

If the upper index bound is infinite, \((a_m)\) fulfills the premises of theorem 5.4, and we are finished.

Assume ad absurdum that the maximal index is \( N \). We shall show how to choose \( a_{N+1} \), giving a contradiction.

For each \( m \in \mathbb{Z}^+, m > N \) define \( A_m = \{b \in \mathbb{R}^{+V_m}\mid a_N = \prod_{i=N+1}^{m}(M^i)^T b\} \). Each \( A_m \) is a nonempty closed set by the above. Any coordinate of any element must be \( \leq a_0 \), and so each \( A_m \) is compact.
Define \( A'_m = \prod_{i=N+2}^{m}(M^i)^TA_m \). Then \((A'_m)\) is a decreasing sequence of nonempty compact sets, whose intersection must therefore be nonempty. Let \( a_{N+1} \) be an element of this intersection. Then \( a_N = (M^{N+1})^TA_{N+1} \) and for each \( m \in \mathbb{Z}^+, m > N, a_{N+1} = \prod_{i=N+2}^{m}(M^i)^Tb \) for some \( b \in \mathbb{R}^+V_m \), which shows that \( a_{N+1} \) can be added to the sequence.

5.3 Measures as positive states on \( K^0(X, T) \)

**Definition 5.3** Assume \( G \) is an Abelian group. A state on \( G \) is an additive function \( s : G \to \mathbb{R} \). If \( G \) is an ordered Abelian group a positive state on \( G \) is a state that maps positive group elements into \( \mathbb{R}^+ \).

Equivalently, a positive state on \( G \) is a positive homomorphism from \( G \) to \((\mathbb{R}, \mathbb{R}^+)\).

**Theorem 5.7** Let \((X, T)\) be a Cantor system. There is a bijective correspondence between \( T \)-invariant Borel measures \( \mu \) on \( X \) and positive states \( s \) on \( K^0(X, T) \), given by \( s([f]) = \int f \, d\mu, f \in C(X, \mathbb{Z}). \)

**Proof:** By passing to a conjugate system there is no loss of generality in assuming that \((X, T)\) is generated by a simple ordered Bratteli diagram \((V, E, <)\).
Let \( f \in C(X, T) \) and let \( \mu \) be a \( T \)-invariant Borel measure on \( X \). Let \( n \geq \text{lev}(f) \). Then
\[
\int f \, d\mu = \sum_{b \in B_n} f(b) \mu(b) = \sum_{v \in V_n} \sum_{p \in P(v, v')} f(p) \mu(p) = \sum_{v \in V_n} \sum_{p \in P(v, v')} f(p) \mu_v
\]
\[
= \sum_{v \in V_n} f_v \mu_v = \langle \vec{f}_n, \vec{\mu}_n \rangle.
\]

From this it is clear that the value of the integral depends only on the equivalence class of \( f \) in \( K^0(X, T) \). Moreover, the integral is additive in \( f \), and positive functions have positive integrals. Thus each \( \mu \) gives a positive state \( s \) on \( K^0(X, T) \) by
\[
s([f]) = \int f \, d\mu, f \in C(X, \mathbb{Z}).
\]

On the other hand, if \( s \) is a positive state on \( K^0(X, T) \), then \( s \circ \gamma_n, n \in \mathbb{Z}^+ \) is a positive state on \( \mathbb{Z}^V_n \) for each \( n \in \mathbb{Z}^+ \).

We write \( s(\gamma_n(x)) = \langle x, a_n \rangle = \sum_{v \in V_n} (a_n)_v x_v, x \in \mathbb{Z}^V_n \) where \( (a_n)_v = s(\gamma_n(\delta_v)) \geq 0, v \in V_n \). Then for \( x \in \mathbb{Z}^V_n \),
\[
(a_n)^T x = \langle x, a_n \rangle = s(\gamma_n(x)) = s(\gamma_{n+1}(\gamma_n^n(x))) = \langle \gamma_{n+1}(x), a_{n+1} \rangle = (a_{n+1})^T M^{n+1} x
\]
and so \( a_n = (M^{n+1})^T a_{n+1} \).

By theorem 5.4 there is then a unique \( T \)-invariant Borel measure \( \mu \) on \( X \) such that \( a_n = \vec{\mu}_n \) for each \( n \in \mathbb{Z}^+ \).

If \( f \in C(X, \mathbb{Z}), \text{lev}(f) = n \) then as above
\[
\int f \, d\mu = \langle \vec{f}_n, \vec{\mu}_n \rangle = \langle \vec{f}_n, a_n \rangle = (a_n)^T \vec{f}_n = s(\gamma_n(\vec{f}_n)) = s([f]).
\]

\[\square\]

### 5.4 Characterisation of positive \( K^0 \) elements by integrals

Let \((X, T)\) be a Cantor System.

**Theorem 5.8** For \( f \in C(X, \mathbb{Z}) \) the following are equivalent:

1. \( [f] > [0] \)
2. \( \int fd\mu > 0 \) for all \( T \)-invariant probability measures \( \mu \).

**Proof:** By passing to a conjugate system we may assume without loss of generality that \((X, T)\) is generated by a simple ordered Bratteli diagram \((V, E, <)\).

(\(\psi\)) This follows since every clopen nonempty set must have non-zero measure: If \([f] = [g]\) where \( g \geq 0, g \neq 0 \) then \( \int f \, d\mu = \int g \, d\mu = \sum_{b \in B_{\text{lev}(g)}} g(b) \mu(b) > 0 \).

32
Theorem 5.6 gives that the set \( \{ \bar{\mu}_{\text{lev}}(f) | \mu \in M(X, T), \mu(X) = 1 \} \) is nonempty compact. Therefore \( \int f \, d\mu = \langle f_{\text{lev}}(f), \bar{\mu}_{\text{lev}}(f) \rangle \) assumes a minimal value, call this \( \epsilon \).

We may then similarly to the proof of theorem 5.6 write:

\[
\emptyset = \bigcap_{m=\text{lev}(f)}^{\infty} \{ b \in \mathbb{R}^{+V_{\text{lev}(f)}} | b = \prod_{i=\text{lev}(f)+1}^{m} (M_i)^T c, c \in \mathbb{R}^{+V_m} \} \bigcap \{ b \in \mathbb{R}^{+V_{\text{lev}(f)}} | \prod_{i=1}^{\text{lev}(f)} (M_i)^T b = 1 \}
\bigcap \{ b \in \mathbb{R}^{+V_{\text{lev}(f)}} | \langle f_{\text{lev}}(f), b \rangle \leq \epsilon/2 \}.
\]

There must then exist an \( N \in \mathbb{Z}^+, N \geq \text{lev}(f) \) such that the compact set

\[
\{ b \in \mathbb{R}^{+V_{\text{lev}(f)}} | b = \prod_{i=\text{lev}(f)+1}^{N} (M_i)^T c, c \in \mathbb{R}^{+V_N} \} \bigcap \{ b \in \mathbb{R}^{+V_{\text{lev}(f)}} | \prod_{i=1}^{\text{lev}(f)} (M_i)^T b = 1 \}
\bigcap \{ b \in \mathbb{R}^{+V_{\text{lev}(f)}} | \langle f_{\text{lev}}(f), b \rangle \leq \epsilon/2 \}
= \emptyset.
\]

But this implies that \( \langle f_N, b \rangle > 0 \) for all \( b \in \mathbb{R}^{+V_N} \setminus \{(0)\} \).

Obviously then \( f_N \in \mathbb{Z}^{+V_N} \setminus \{(0)\} \).

\[ \square \]

5.5 Example of measure with only trivial invariant transformation

**Definition 5.4** Two Cantor systems are orbit equivalent if there is a homeomorphism between their Cantor sets preserving the property of two points having the same orbit.

In [3] a proof is given that two cantor systems are orbit equivalent if and only if they have the same invariant measures in the following sense: There exist a homeomorphism between the spaces transferring the measures of each system onto the other.

It is a fact that there exist Cantor systems (and other dynamical systems) such that they have only one invariant measure, up to a constant factor. Such systems are known as uniquely ergodic.

Given the above, it would be interesting to classify the dynamical systems having a given measure as their unique invariant measure, as they would all be orbit equivalent.

We may ask whether every non-atomic probability measure on a Cantor set is the unique invariant measure of some dynamical system. The goal of this section is to construct a thorough counterexample to this.

Consider a Cantor set \( X \) equipped with a standard basis of clopen sets, arranged in a binary tree:
We number the basis elements in levels as above. Fix a transcendental number \( a > 0 \).

Define a function \( \mu \) from the basis to \([0, 1]\) by

\[ \mu(B_{n,m}) = \frac{a^{m/(2^n - 1)}}{\sum_{i=0}^{2^n-1} a^{i/(2^n-1)}}. \]

We note that \( \mu \) is additive. We can therefore similarly to the proof of theorem 5.4 extend \( \mu \) to a function \( \hat{\mu} \) on all clopen sets which is still additive. By theorem 5.2 we can then extend \( \mu \) to a complete borel measure.

Now fix two clopen sets \( M \) and \( N \). Both must be a finite union of basis sets; find the maximal level \( n \) of these basis sets; then \( M \) and \( N \) is each a finite disjoint union of level \( n \) basis sets. Then if \( M = B_{n,m_1} \cup B_{n,m_2} \cup \cdots \cup B_{n,m_k}, m_1 < m_2 < \ldots < m_k \) and \( N = B_{n,n_1} \cup B_{n,n_2} \cup \cdots \cup B_{n,n_l}, n_1 < n_2 < \ldots < n_l \) we have

\[ \mu(M) = \frac{\sum_{i=1}^{k} a^{m_i/(2^n - 1)}}{\sum_{i=0}^{2^n-1} a^{i/(2^n-1)}}, \mu(N) = \frac{\sum_{i=1}^{l} a^{n_i/(2^n - 1)}}{\sum_{i=0}^{2^n-1} a^{i/(2^n-1)}}. \]

thus if \( \mu(M) = \mu(N) \), we have

\[ \sum_{i=1}^{k} a^{m_i/(2^n - 1)} = \sum_{i=1}^{l} a^{n_i/(2^n - 1)}. \]

Now since \( a \) is transcendental, \( a^{1/2} \) is too (since the algebraic numbers form a field) and by induction so is \( b = a^{1/(2^n - 1)} \). Thus

\[ \sum_{i=1}^{k} b^{m_i} = \sum_{i=1}^{l} b^{n_i}. \]

but this contradicts transcendentality unless \( (m_i) = (n_i) \), thus \( M = N \). So we have shown that no two disjoint clopen sets have the same measure. Imagine then a measure-preserving homeomorphism \( T : X \to X \); it must preserve each clopen set, and since every one point set \( x \) is closed and thus the intersection of the clopen basis sets containing it, \( T \) must fix all points and thus \( T = \text{Id}_X \).
Index

< 7
[•] 18
[• • ] 7
∞ 6

algebra 26
associated dimension group of a Bratteli diagram 17
Borel algebra 26
Borel measure 26
Borel set 26
Bratteli diagram 6
$B(X)$ 26
$B(X, T)$ 18
χ • 18
Cantor set 5
Cantor system 5
cardinality 7
characteristic function 18
clopen set 5
completely connected Bratteli diagram 9
conjugacy 4
$C(X, Z)$ 18
cylinder set 11
δ • 15
dimension group 17
direct limit of a directed sequence 15
directed sequence of ordered Abelian groups 15
dynamical system 4
$E$ 6
edge of Bratteli diagram 6
edge set 6
$E_i$ 6
equivalence class 18
finite measure 26

$f_n$ 19
full group on $(X, T)$ 24
$f_v$ 19
$G^+$ 14
$\gamma_m^n$ 15
$\gamma_m^n$ 17
$\gamma_n$ 15
hereditary subgroup 17
Id • 4
incidence matrix 7
infinite path 9
integer 4
isomorphic ordered Abelian groups 14
iteration 4
$K^0(X, T)$ 18
lev 19
level of a basis element 11
level of a vertex 6
level of an edge 6
level of function 19
$M^\bullet$ 7
matrix product 7
maximal path 7
measure 26
minimal dynamical system 5
minimal path 7
$\bar{\mu}_n$ 27
$\mu_v$ 27
$M(X, T)$ 27
negative element of an ordered Abelian group 14
negative orbit 4
nonnegative integer 4
nonnegative real number 4
nontrivial Bratteli diagram 9
<table>
<thead>
<tr>
<th>Term</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>orbit equivalence</td>
<td>33</td>
</tr>
<tr>
<td>orbit of a point</td>
<td>4</td>
</tr>
<tr>
<td>orbit of a set</td>
<td>4</td>
</tr>
<tr>
<td>order ideal</td>
<td>17</td>
</tr>
<tr>
<td>ordered Abelian group</td>
<td>14</td>
</tr>
<tr>
<td>ordered Bratteli diagram</td>
<td>7</td>
</tr>
<tr>
<td>ordering of edges</td>
<td>7</td>
</tr>
<tr>
<td>ordering of path sets</td>
<td>7</td>
</tr>
<tr>
<td>$P(\bullet, \bullet)$</td>
<td>7</td>
</tr>
<tr>
<td>$\Pi$</td>
<td>7</td>
</tr>
<tr>
<td>path</td>
<td>6</td>
</tr>
<tr>
<td>positive element of an ordered Abelian group</td>
<td>14</td>
</tr>
<tr>
<td>positive homomorphism</td>
<td>14</td>
</tr>
<tr>
<td>positive orbit of a point</td>
<td>4</td>
</tr>
<tr>
<td>positive orbit of a set</td>
<td>4</td>
</tr>
<tr>
<td>positive state on a group</td>
<td>31</td>
</tr>
<tr>
<td>pred</td>
<td>7</td>
</tr>
<tr>
<td>predecessor of a path</td>
<td>7</td>
</tr>
<tr>
<td>probability measure</td>
<td>26</td>
</tr>
<tr>
<td>$\mathbb{R}$</td>
<td>4</td>
</tr>
<tr>
<td>$\mathbb{R}^+$</td>
<td>4</td>
</tr>
<tr>
<td>$r$</td>
<td>6</td>
</tr>
<tr>
<td>range map</td>
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Bibliography


